MTH 310.003 | **Review Exercises** | **Fall 2018**

Review Exercises

1. Consider the field $\mathbb{Z}_{17} = \mathbb{Z}/(17)$.

   (a) Find the reciprocals $1^{-1}, 2^{-1}, \ldots, 16^{-1} \in \mathbb{Z}_{17}$.

   (b) Make a table of squares in $\mathbb{Z}_{17}$ in reduced form (e.g. $6^2 = 36 = 2$, so $\sqrt{2} = 6 \in \mathbb{Z}_{17}$. Explain the symmetry of this table: it begins 1, 4, 9, ..., and ends ..., 9, 4, 1.

   (c) Solve the equation $2x^2 + 4x + 1 = 0$ for $x \in \mathbb{Z}_{17}$.

2. A field K with 8 elements

   (a) Construct such a field K as an extension field of $\mathbb{Z}_2$. Take an irreducible polyomial $p(x)$ of degree 3, and let $K = \mathbb{Z}_2/(p(x))$. Write the 8 elements as standard forms in the compact notation (α instead of $[x]$, no brackets on coefficients).

   (b) Find the reciprocal of each element in K.

   (c) Factor $p(y)$ completely into linear factors in $K[y]$, allowing coefficients in K. Hint: One root of $p(y)$ is $y = \alpha$. Check that $y = \alpha^2$ is another root.

3. It is difficult to write down a real solution to $x^3 + x + 1 = 0$. Assuming α is such a solution, consider the ring:

$$K = \{a_0 + a_1\alpha + \cdots + a_n\alpha^n \text{ for } a_i \in \mathbb{Q},\ n \geq 0\}.$$

   (a) Consider the homomorphism $\phi : \mathbb{Q}[x] \to K$ given by $\phi(f(x)) = f(\alpha)$. Find the kernel of ϕ. Hint: It is clear that $p(x) = x^3 + x + 1 \in \mathrm{Ker}(\phi)$, so the principal ideal $(p(x)) \subset \mathrm{Ker}(\phi)$. Now note that $p(x)$ is irreducible in $\mathbb{Q}[x]$, so $(p(x))$ is a maximal ideal. Could there be any more elements of $\mathrm{Ker}(\phi)$?

   (b) Use a theorem to that conclude that $K \cong \mathbb{Q}[x]/(x^3 + x + 1)$, and that K is a field.

   (c) Assuming part (b), show that any element of K can be written in the form $a_0 + a_1\alpha + a_2\alpha^2$.

   (d) Find the reciprocal $\frac{1}{\beta} = b_0 + b_1\alpha + b_2\alpha^2$ of the element $\beta = 1 + \alpha^2$ by the Euclidean Algorithm applied to $x^2 + 1$ and $x^3 + x + 1$.